Imperfect Information Game for a Simple Pursuit-Evasion Problem

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Differential Games for pursuit evasion problems have been investigated for many years. Differential games, with linear state equations and quadratic cost functions, are called Linear Quadratic Differential Game (LQDG). In these games, one defines two players a pursuer and an evader, where the former aims to minimize and the latter aims to maximize the same cost function (zero-sum games). The main advantage in using the LQDG formulation is that one gets Proportional Navigation (PN) like solutions with continuous control functions. One approach which plays a main role in the LQDG literature is Disturbance Attenuation (DA), whereby target maneuvers and measurement error are considered as external disturbances. In this approach, a general representation of the input-output relationship between disturbances and output performance measure is the DA function (or ratio). This function is bounded by the control. This work revisits and elaborates upon this approach. We introduce the equivalence between two main implementations of the DA control. We then study a representative case, a "Simple Pursuit Evasion Problem", with perfect and imperfect information patterns. By the derivation of the analytical solution for this game, and by running some numerical simulations, we develop the optimal solution based on the critical values of the DA ratio. The qualitative and quantitative properties of the Simple Pursuit Evasion Problem, based on the critical DA ratio, are studied by extensive numerical simulations, and are shown to be different from the fixed DA ratio solutions.

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I. Introduction

Pursuit-evasion differential games have attracted considerable attention since the seminal works of [1, 2]. A class of differential games, with a couple of players driving linear state equations both affecting a quadratic cost function, is called Linear Quadratic Differential Game (LQDG). In these games, the pursuer tries to minimize a quadratic cost function, whereas the evader tries to maximize the same cost function (zero-sum games). The cost function includes weights on the squared miss-distance, the control efforts of both players, as well as occasionally trajectory shaping terms. The main LQDG formulation leads to a derivation of popular guidance laws such as Proportional Navigation (PN), Optimal Rendezvous (OR), etc. A problem which is closely related to the LQDG problem is the one of Disturbance Attenuation (DA), where pursuer actions are considered to be control actions, whereas all external actions such as target maneuvers and measurement errors, are considered to be disturbances [3, 4]. In fact, the DA problem is just one side of the double inequality forming the saddle-point relation and leads to the H - infinity problem. DA problems can either deal with perfect information patterns, where both pursuer and evader share perfect information regarding the full state-vector, or imperfect information formulation, where both players have access only to noisy measurements of a linear combination of the state-vector [5, 6]. The present paper, revisits DA problems in the latter formulation, and performs detailed analyses for a simple pursuit-evasion example which provide insight to the interplay between the control and the estimation parts of the pursuer strategy.

We firstly introduce the equivalence between two main implementations of the DA control, one formulated by Speyer and Jacobson [3] and the other by Green and Limebeer [4]. Secondly, we introduce and discuss a representative case study of a Simple pursuit evasion Problem, with perfect and imperfect information patterns. We derive the optimal solution, and propose an approximate solution. We then present some numerical results, for the Simple Pursuit Evasion Problem using critical and non-critical values of the DA solution.

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II. The Equivalence between Two Implementations of the Pursuer Strategy

Consider the following system as can be found also in [3, 7]

$$\dot{x} = Ax + B_1 w + B_2 u \quad , x(0) = x_0 \tag{1}$$

$$y = C_2 x + D_{21} w \tag{2}$$

$$z = C_1 x + D_{12} u \tag{3}$$

Where$x \in R^n$ is the state vector, $w \in R^q$ is an exogenous disturbance, $u \in R^s$is the control input signal, $x_0$is an unknown initial state, $y \in R^r$ is the output measurement. The matrices $A, B_1, B_2, C_1, C_2, D_{12}, D_{21}$

are constant matrices of the appropriate dimensions, satisfying

$$D_{21}^T B_1 = 0,\ D_{21} D_{21}^T = I,\ D_{12}^T C_1 = 0,\ D_{12} D_{12}^T = I \tag{4}$$

where I stands for the identity matrix. Consider also the following cost function

$$J = x^T(t_f) X_f x(t_f) - \gamma^2 x_0{}^T Y_0{}^{-1} x_0 + \int_0^{t_f} \left(z^T z - \gamma^2 w^T w \right) dt \tag{5}$$

which is to be respectively minimized and maximized by u and w. In connection with the Linear Quadratic Differential Game (LQDG) problem, the following couple of modified Differential Riccati Equations (DRE) plays important role

$$-\dot{X} = \hat{A}_2^T X + XA + C_1{}^T C_1 \tag{6}$$

$$\dot{Y} = \hat{A}_1 Y + YA^T + B_1 B_1{}^T \tag{7}$$

where

$$\hat{A}_1 \triangleq A + \gamma^{-2} Y C_1{}^T C_1 - Y C_2{}^T C_2 \tag{8}$$

$$\hat{A}_2 \triangleq A + \gamma^{-2} B_1 B_1{}^T X - B_2 B_2{}^T X \tag{9}$$

Also a third DRE, the solution of which is denominated by Z is known to be related to X and Y, by

$$Z = \left(I - \gamma^{-2} Y X \right)^{-1} Y \tag{10}$$

The LQDG literature presents the following solution to the above measurement feedback problem where w can take any full information strategy (i.e. with access to-x), whereas u has access only to the noisy measurement y. The following solution has been found in [3]

$$\dot{\hat{x}}_1 = \bar{A}_1 \hat{x}_1 + \bar{B}_1 y \tag{11}$$

$$u = \bar{C}_1 \hat{x}_1 \tag{12}$$

where

$$\bar{A}_1 = \hat{A}_1 - B_2 B_2{}^T X \left(I - \gamma^{-2} Y X \right)^{-1} \tag{13}$$

$$\bar{B}_1 = Y C_2{}^T \tag{14}$$

$$\bar{C}_1 = -{B_2}^T X \big(I - \gamma^{-2} Y X\big)^{-1} \tag{15}$$

Another solution appears in [4] which generally serves the control community

$$\dot{\hat{x}}_2 = \bar{A}_2 \hat{x}_2 + \bar{B}_2 y \tag{16}$$

$$u = \bar{C}_2 \hat{x}_2 \tag{17}$$

where

$$\bar{A}_2 = \hat{A}_2 - Z {C_2}^T C_2 \tag{18}$$

$$\bar{B}_2 = Z {C_2}^T \tag{19}$$

$$\bar{C}_2 = -{B_2}^T X \tag{20}$$

In [3] the two solutions have been shown to coincide in the finite time varying case. Our aim is to show that, as could be expected, the solutions are equivalent. We use here a somewhat different approach which applies a similarity transformation between these two implementations. To this end, consider $T > 0$, such that

$$\hat{x}_2 = T^{-1} \hat{x}_1 \tag{21}$$

We readily obtain using the following identity, (22) that the two implementations are equivalent.

$$\frac{dT^{-1}}{dt} \equiv -T^{-1} \dot{T} T^{-1} \tag{22}$$

Therefore, we seek for $T > 0$, so that

$$\bar{A}_1 = T \bar{A}_2 T^{-1} + \dot{T} T^{-1} \tag{23}$$

$$\bar{B}_1 = T \bar{B}_2 \tag{24}$$

$$\bar{C}_1 = \bar{C}_2 T^{-1} \tag{25}$$

We next intend to show that $T = I - \gamma^{-2} Y X$ satisfies the above relation. Indeed,

$$\bar{C}_1 = -{B_2}^T X \big(I - \gamma^{-2} Y X\big)^{-1} = \bar{C}_2 T^{-1} \tag{26}$$

$$\bar{B}_1 = Y{C_2}^T = TZ{C_2}^T = \left(I - \gamma^{-2}YX\right)Z{C_2}^T = T\bar{B}_2 \tag{27}$$

It, therefore, remains to establish the relation between $\bar{A}_1$ and $\bar{A}_2$. To this end, note that using the above definition of T we have

$$\bar{A}_1 = \hat{A}_1 - B_2{B_2}^T XT^{-1} \tag{28}$$

$$\bar{A}_2 = \hat{A}_2 - Z{C_2}^T C_2 \tag{29}$$

From (23) we have to show that

$$\bar{A}_1 T = T\bar{A}_2 + \dot{T} \tag{30}$$

In other words,

$$\hat{A}_1 T - B_2{B_2}^T X = T\hat{A}_2 - Y{C_2}^T C_2 + \dot{T} \tag{31}$$

where we have used the relation $TZ{=}Y$. The expression for transformation derivative is given by

$$\dot{T} = -\gamma^{-2}\dot{Y}X - \gamma^{-2}Y\dot{X} \tag{32}$$

Substitute (32) in (31) and using the DRE's of (6) , (7) yield the following:

$$L \triangleq \hat{A}_1 - \gamma^{-2}\left(\dot{Y} - B_1{B_1}^T - YA^T\right)X - B_2{B_2}^T X \tag{33}$$

$$R \triangleq \hat{A}_2 - \gamma^{-2}Y\left(-\dot{X} - {C_1}^T C_1 - A^T X\right) - Y{C_2}^T C_2 - \gamma^{-2}\dot{Y}X - \gamma^{-2}Y\dot{X} \tag{34}$$

Finally, substitute $\hat{A}_1$, $\hat{A}_2$ and collecting terms, we readily find that $L{=}R$ as required, thus completing the proof of similarity based equivalence.

Note: One may choose the initial condition, and use a trivial transformation between the two implementations, in order to achieve this equivalency. As a result, we can see easily the equivalence between the two controls

$$u_1 = \bar{C}_1\hat{x}_1 = -{B_2}^T X\left(I - \gamma^{-2}YX\right)^{-1}\hat{x}_1 \tag{35}$$

$$u_2 = \bar{C}_2\hat{x}_2 = -{B_2}^T X\hat{x}_2 \tag{36}$$

$$\hat{x}_2 = T^{-1}\hat{x}_1 \tag{37}$$

$$T = I - \gamma^{-2} Y X \tag{38}$$

$$u_2 = -{B_2}^T X T^{-1} \hat{x}_1 = -{B_2}^T X \left(I - \gamma^{-2} Y X\right)^{-1} \hat{x}_1 \tag{39}$$

$$u_1 \equiv u_2 \tag{40}$$

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III. Special Case-The Simple Pursuit Evasion Problem

Consider two players, A and B, where A (the pursuer) wants to hit B (the evader). In order to accomplish this mission, A directly controls its heading angle α, trying to navigate toward B. On the other hand, B tries to evade from A, and does that by directly controlling its heading angle β.

Assumptions:

-Two dimensional problem.

-The players have constant velocities: V_A , V_B.

-The pursuer is faster than the evader: $V_A > V_B$.

-Both players have direct control over their heading angles $\alpha \ll 1$ and $\beta \ll 1$ (fast angle control which can be neglected due to small deviations).

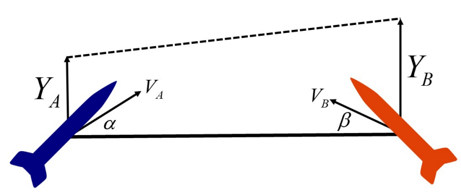

Fig. 1 The Simple PursuitEvasion Problem, Geometry Description

The equations of motion are

$$\begin{aligned} & x \triangleq Y_B - Y_A \\ & x(0) = 0 \end{aligned} \tag{41}$$

$$\dot{x} = V_B \sin\beta - V_A \sin\alpha \triangleq w + u \tag{42}$$

$$z = x + v \tag{43}$$

Here $x \in R^1$ is the "relative separation", $z \in R^1$ is the measurement, and $v \in R^1$ is an additive noise. We can formulate the problem as Min-Max problem, where A aims to minimize the relative separation at the terminal time, whereas B wants to maximize it.

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3.1 Optimal Solution- Perfect Information Game

Consider first the perfect information game-formulation where there is zero measurement noise. From (42) and (6) we get

$$\dot{x} = w + u, x(0) = 0 \tag{44}$$

$$\min_u \max_w J(u, w) = \frac{b}{2} x^2(t_f) + \frac{1}{2} \int_0^{t_f} \left(u^2(t) - \gamma^2 w^2(t) \right) dt \tag{45}$$

$$\dot{X}(t) = \left(1 - \gamma^{-2}\right) X^2(t) \quad , X(t_f) = b \tag{46}$$

The optimal strategies are given by [8]

$$u^*(t) = -X(t) x(t) \tag{47}$$

$$w^*(t) = \gamma^{-2} X(t) x(t) \tag{48}$$

$$X(t) = \frac{1}{(1 - \gamma^{-2})(t_f - t) + 1/b} \tag{49}$$

Note that $\gamma > 1$ guarantees a positive definite $X(t)\ \forall t$. One obtains the linear control feedbacks (players' strategies) as follows

$$u^* = -\frac{1}{(1 - \gamma^{-2})(t_f - t) + 1/b} x(t) \tag{50}$$

$$w^* = \gamma^{-2} \frac{1}{(1 - \gamma^{-2})(t_f - t) + 1/b} x(t) \tag{51}$$

Note that by letting $b \to \infty, \gamma \to \infty$, we get a simple Collision Course Guidance (CCG) Law

$$u^* = -\frac{1}{(t_f - t)} x(t) \tag{52}$$

$$w^* = 0 \tag{53}$$

3.2 Optimal Solution- Imperfect Information Game

Consider imperfect information game such that there is a non-zero noise, v, added the pursuer's measurements. $x(t_0)$ is an unknown initial state. From (5) we get

$$\max_{w}\min_{u} J = -\frac{1}{2}\gamma^2 {Y_0}^{-1} {x_0}^2 + \frac{1}{2} b{x_f}^2 + \frac{1}{2}\int_{t_0}^{t_f} \left(u^2 - \gamma^2\left(w^2 + V^{-1}v^2\right)\right) dt \tag{54}$$

$$\dot{x} = u + w \tag{55}$$

$$z = x + v \tag{56}$$

Notice that by adding a weight V^{-1} to the quadratic term of v (in Eq. (5) the noise was normalized by V^{-1}). From (12) and (15), the pursuer optimal control is given by

$$u^* = -\frac{X}{1-\gamma^{-2}YX}\hat{x} \tag{57}$$

where we denote the estimated state feedback gain by

$$\Lambda \triangleq -\frac{X}{1-\gamma^{-2}YX} = -\frac{1}{X^{-1}-\gamma^{-2}Y} \tag{58}$$

and where (6) for our case now is given by

$$X\left(t_f\right) = b \tag{59}$$

$$\dot{X} = X^2\left(1-\gamma^{-2}\right) \tag{60}$$

$$X\left(t\right) = \frac{1}{\left(1-\gamma^{-2}\right)\left(t_f - t\right) + 1/b} \tag{61}$$

where the estimation Riccati equation for *Y(t)* in our case then reads

$$\dot{Y} = AY + YA^T + DD^T - YH^TV^{-1}HY \tag{62}$$

where

$$A = 0,\ D = 1,\ H = 1 \tag{63}$$

DRE for *Y(t)* can be written simplicity as

$$\begin{aligned}&\dot{Y} = 1 - Y^2V^{-1}\\ &Y\left(0\right) = Y_0\end{aligned} \tag{64}$$

The steady state value is given by

$$\begin{aligned} & 0 = 1 - Y^2 V^{-1} \\ & \rightarrow Y^2 = V \end{aligned} \tag{65}$$

where the solution to the DRE can be found in [9]

$$Y(t) = \sqrt{V} + \frac{2\sqrt{V}}{\left(\dfrac{Y_0 + \sqrt{V}}{Y_0 - \sqrt{V}}\right) e^{\dfrac{2t}{\sqrt{V}}} - 1} \tag{66}$$

One can notice that in the steady state, $Y_{ss} = \sqrt{V}$. This result coincides with the Kalman Filter (KF) solution. Finally, the gain is given by

$$\Lambda = \frac{-1}{(1 - \gamma^{-2})(t_f - t) + 1/b - \gamma^{-2}\sqrt{V} - \gamma^{-2} \cdot \dfrac{2\sqrt{V}}{\left(\dfrac{Y_0 + \sqrt{V}}{Y_0 - \sqrt{V}}\right) e^{\dfrac{2t}{\sqrt{V}}} - 1}} \tag{67}$$

where

$$u^* = \Lambda \hat{x}, \hat{x}(t_0) = 0 \tag{68}$$

$$\dot{\hat{x}}(t) = \Lambda \hat{x} + Y V^{-1} (z - \hat{x}) \tag{69}$$

For optimality, we demand the following three conditions:

1. Solution to DRE, *Y(t)*, exists $\forall t \in [t_0, t_f]$.
2. Solution to DRE, X*(t)*, exists $\forall t \in [t_0, t_f]$.
3. The Spectral Radius Condition (SRC): $1 - \gamma^{-2} Y X > 0 \, \forall t \in [t_0, t_f]$.

For the described case, it results in the following inequalities:

$$\sqrt{V} + \frac{2\sqrt{V}}{\left(\dfrac{Y_0 + \sqrt{V}}{Y_0 - \sqrt{V}}\right) e^{\dfrac{2t}{\sqrt{V}}} - 1} > 0 \tag{70}$$

$$\left(1 - \gamma^{-2}\right)(t_f - t) + 1/b > 0 \tag{71}$$

$$\left(1 - \gamma^{-2}\right)(t_f - t) + 1/b - \gamma^{-2}\sqrt{V} - \gamma^{-2} \cdot \frac{2\sqrt{V}}{\left(\dfrac{Y_0 + \sqrt{V}}{Y_0 - \sqrt{V}}\right) e^{\dfrac{2t}{\sqrt{V}}} - 1} > 0 \tag{72}$$

The conditions can be summarized by the following expression

$$\left(1-\gamma^{-2}\right)\left(t_f-t\right)+1/b>\gamma^{-2}\sqrt{V}+\gamma^{-2}\cdot\frac{2\sqrt{V}}{\left(\frac{Y_0+\sqrt{V}}{Y_0-\sqrt{V}}\right)e^{\frac{2t}{\sqrt{V}}}-1}>0 \tag{73}$$

We can get a lower bound for γ^2 as follows

$$\gamma^2>\frac{\sqrt{V}+\left(t_f-t\right)}{1/b+\left(t_f-t\right)}+\frac{2\sqrt{V}}{\left(\frac{Y_0+\sqrt{V}}{Y_0-\sqrt{V}}\right)e^{\frac{2t}{\sqrt{V}}}-1}\cdot\left(\frac{1}{1/b+\left(t_f-t\right)}\right) \tag{74}$$

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IV. Approximation using Steady State LKF

Taking the steady state solution for *Y(t)*

$$Y_{ss}=\sqrt{V} \tag{75}$$

From that, one can approximate the expression of the gain to be as following:

$$\tilde{\Lambda}=\frac{-1}{\left(1-\gamma^{-2}\right)\left(t_f-t\right)+1/b-\gamma^{-2}\sqrt{V}} \tag{76}$$

This expression holds after a short transient of Y, for sufficiently small V (see eq. (67)). Control and estimate equations are obtained

$$u^*=-\frac{1}{\left(1-\gamma^{-2}\right)\left(t_f-t\right)+1/b-\gamma^{-2}\sqrt{V}}\hat{x} \tag{77}$$

$$\dot{\hat{x}}\left(t\right)=\left(\tilde{\Lambda}-\frac{1}{\sqrt{V}}\right)\hat{x}+\frac{1}{\sqrt{V}}z \tag{78}$$

In Fig. 2, we run two simulations for the same scenario, one with the approximated gain and one with the exact gain. The simulation parameters are

$$b=50, V=1\left[m^2\right], \gamma^2=50, Y_0=10\left[m^2\right]$$

One can observe the fast convergence of the approximate to the exact gain.

We can investigate now the approximate gain expression (76). Let ${\gamma_c}^2$ be the minimum value of γ^2 such that the solution to our problem exists, satisfying the three conditions of the last paragraph. The gain (76) must hold for all $t\in[t_0,t_f]$, and in particular for $t=t_f$. By substituting t_f, the gain reduces to

$$\tilde{\Lambda}\left(t_f\right)=-\frac{1}{1/b-\gamma^{-2}\sqrt{V}} \tag{79}$$

In order to meet the third optimally condition (SRC), the following inequality must hold

$$1/b-\gamma^{-2}\sqrt{V}>0 \tag{80}$$

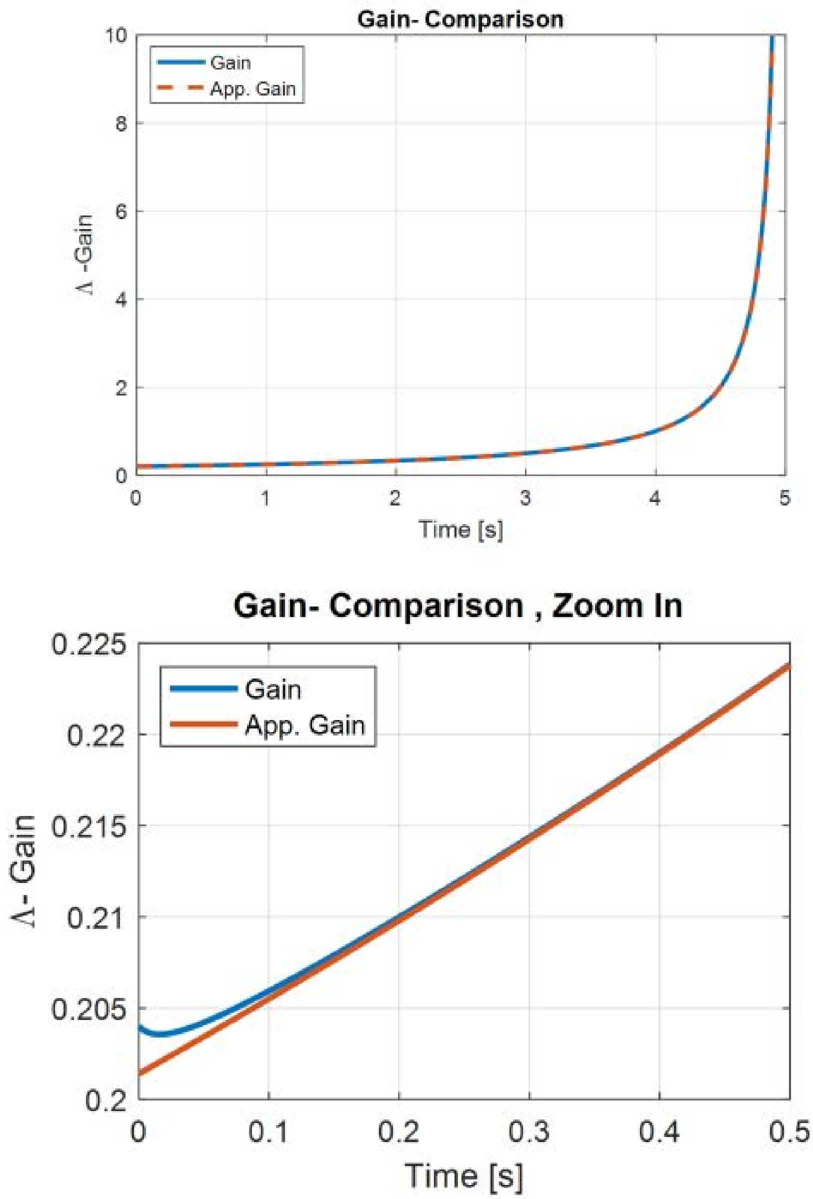

Fig. 2 The Convergence in Order to Justify the Approximated Gain.

Notice that if (80) does not hold, then (67) is also violated (under our approximation) for sufficiently small $(t_f - t)$. From the last inequality we can get a lower bound for γ^2, as follows

$$\gamma^2 > b\sqrt{V} \tag{81}$$

We got the critical value for the DA ratio

$$\left(\gamma_c{}^2\right)_1 \triangleq b\sqrt{V} \tag{82}$$

From the second inequality, we still require

$$\left(\gamma_c{}^2\right)_2 \triangleq 1 \tag{83}$$

A generalization of the last two results yields $\gamma_c{}^2 = \max\left\{b\sqrt{V}, 1\right\}$. Under this approximation, we can examine the gain behavior with respect to the noise magnitude V. From exploring (76), one can notice that as the noise magnitude gets higher, the difference between the positive and negative values in the denominator reduces. As a result, the gain grows. This will always be the case when we fix γ^2 to some sufficiently large value $\gamma^2 > \gamma_c{}^2$, and we keep changing the noise term V (provided that our fixed γ^2 is always greater than $b\sqrt{V}$). On the other hand, using the critical value of $\gamma^2 = \gamma_c{}^2$, readjusting it for each V by (82), one gets

$$\tilde{\Lambda}^{\dagger} \triangleq -\frac{1}{\left(1 - \frac{1}{b\sqrt{V}}\right)(t_f - t)} \tag{84}$$

Notice the opposite behavior of the gain with the noise magnitude: as the noise magnitude gets higher, the gain gets lower. For this case, the optimal control and the estimate equation are given by

$$u^* = \tilde{\Lambda}^{\dagger}\hat{x} = -\frac{1}{\left(1 - \frac{1}{b\sqrt{V}}\right)(t_f - t)}\hat{x} \tag{85}$$

$$\dot{\hat{x}}(t) = \left(\tilde{\Lambda}^{\dagger} - \frac{1}{\sqrt{V}}\right)\hat{x} + \frac{1}{\sqrt{V}}z \tag{86}$$

We may consider the following two limit cases

$$\left|\tilde{\Lambda}\right|^* \triangleq \lim_{b\to\infty}\left|\tilde{\Lambda}\right| = \lim_{V\to\infty}\left|\tilde{\Lambda}\right| = \frac{1}{(t_f - t)} \tag{87}$$

We get that the last gains are equivalent to the perfect information case (CCG).

V. Numerical Results

The SRC entails

$$\Omega \triangleq 1 - \gamma^{-2}Y(t) \cdot X(t) \tag{88}$$

$$\Omega_{\min} = 1 - \gamma_c{}^{-2}Y(t_c) \cdot X(t_c) \tag{89}$$

$$\Omega_{\min} = 0 \Rightarrow 1 - \gamma_c{}^{-2}Y(t_c) \cdot X(t_c) = 0 \tag{90}$$

$$\gamma_c{}^2 = Y(t_c) \cdot X(t_c) \tag{91}$$

In practice, it is not recommended taking the minimum value, and a safety margin from singularity should be considered. We choose $\Omega_{\min} = 0.1$. To demonstrate the results, we run a simulation using a Gaussian White Noise (GWN) with different values for the standard deviation η. The value of V is adjusted to η by $V = \eta^2$. The target performs various maneuvers. The weight of the terminal relative separation (the miss distance) is fixed to b=50. For each case, we will examine two strategies: the first with a fixed γ and the second with a minimal γ (up to $\Omega_{\min} = 0.1$). Typical state history and control gains are shown for each case, for the minimal γ on the left and for a fixed γ on the right. The fixed γ has been chosen as the larger of the two γ 's. Monte Carlo results with 500 runs are given in tables (below the gains) for the main two statistical moments of the miss distance and the control effort. Four cases were simulated for the target speed: Case #1 with constant speed of 3 [m/s], Case #2 with varying speed of 3sin(5t) [m/s], Case #3 is similar to Case #1 but with different noise level and Case #4 with bang-bang maneuver at t=1.5[s] where at the beginning the target stays at rest and then moving with constant speed of 3 [m/s]. The results are given in Tables 1-4 and Figures 3-6. One can observe that, when the control is designed with a fixed value for γ, the gain grows as the noise level grows. However, when we consider the near-critical DA value, the gain is reduced as the noise level grows, which is the more intuitive result. For all case we achieve a lower miss-distance with the near-critical DA values. However, the control effort sometimes increases (e.g. Case 2).

Table 1 Case 1, MC with 500 iterations

$\eta[m]$	γ	$E\left[x\left(t_f\right)\right]$	$\sigma_{x(t_f)}$	$E\left[u_{eff}\right]$	$\sigma_{u_{eff}}$
0.1	2.28	0.34	0.19	23.2	2.49
0.1	3.20	0.46	0.19	27.5	2.21
0.2	3.20	0.63	0.30	23.8	3.77

Table 2 Case 2, MC with 500 iterations

$\eta[m]$	γ	$E\left[x\left(t_f\right)\right]$	$\sigma_{x(t_f)}$	$E\left[u_{eff}\right]$	$\sigma_{u_{eff}}$
0.1	2.28	0.27	0.17	3.38	1.56
0.1	3.20	0.33	0.17	1.13	0.97
0.2	3.20	0.49	0.27	1.94	1.41

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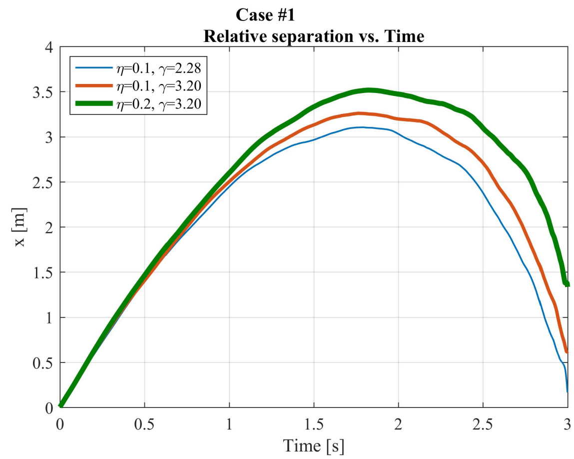

Fig. 3 Case #1

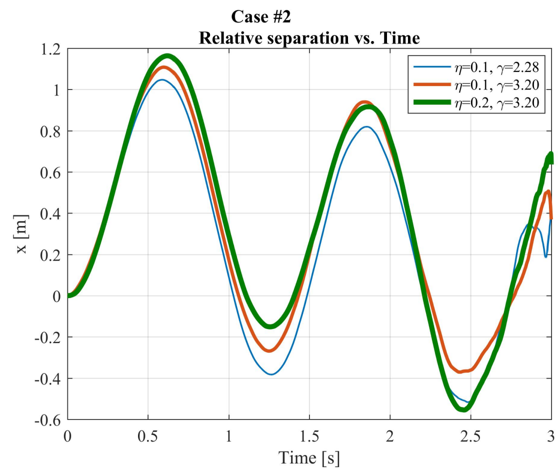

Fig. 4 Case #2

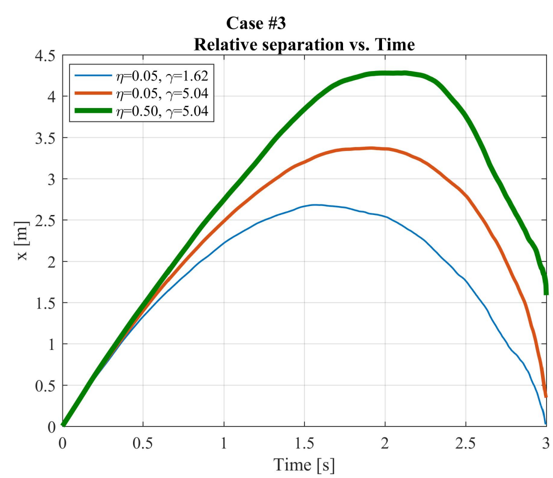

Fig. 5 Case #3

Table 3 Case 3, MC with 500 iterations

$\eta[m]$	γ	$E\left[x\left(t_f\right)\right]$	$\sigma_{x(t_f)}$	$E\left[u_{eff}\right]$	$\sigma_{u_{eff}}$
0.05	1.62	0.18	0.12	23.0	1.37
0.05	5.04	0.44	0.11	22.7	1.25
0.50	5.04	1.48	0.45	26.1	5.15

Table 4 Case 4, MC with 500 iterations

$\eta[m]$	γ	$E\left[x\left(t_f\right)\right]$	$\sigma_{x(t_f)}$	$E\left[u_{eff}\right]$	$\sigma_{u_{eff}}$
0.05	1.62	0.18	0.12	10.7	1.38
0.05	5.04	0.38	0.12	13.0	1.18
0.50	5.04	1.49	0.51	4.72	4.12

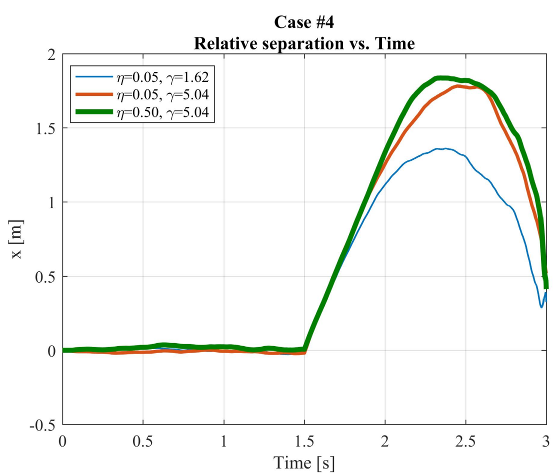

Fig. 6 Case #4

VI. Conclusion

The problem of DA with imperfect information pattern has been revisited. First the equivalence between the two main DA control solution formulations was established for the finite-time horizon case. Then, a representative case were introduced and solved in closed form. Numerical simulations demonstrate the advantage of using the critical value of DA ratio over a fixed DA ratio in obtaining smaller miss distances at the possible expanse of larger control effort. Thus, when the control is cheap one should use the minimal values, whereas for limited control systems, higher DA ratio may be used. Although we focused in simple guidance problems, these problems seem to capture some of the main characteristics of problems of higher complexity.

References

[1] Ho, A. E. Y.-C., *Applied Optimal Control*, John Wiley & Sons, 1975.

[2] Isaacs, . R., *Differential Games: A Mathematical Theory with Applica-tion to Warfare and Pursuit, Control and Optimization*, John Wiley & Sons, 1965.

[3] Speyer, . J. L. and H, D., "Jacobson,"," *Primer on Optimal Control Theory"*, 2010, pp. 231–250.

[4] Green, . M. and Limebeer, D. J., *Linear Robust Control*, Prentice-Hall, 1995.

[5] *Jason L.Speyer, "An Adaptive Terminal Guidance Scheme Based on an Exponential Cost Criterion with Application to Homing Missile Guid-ance"*, IEEE, 1976.

[6] Z, . J., *Ben-Asher and Jason L.Speyer, "Game in Aerospace: Homing Missile Guidance"*, Springer, 2017.

[7] Gershon, U. S. . E., *Isaac Yaesh, "H infinity Control and Estimation of State-Multiplicative Linear Systems"*, Springer, 2005.

[8] Z, . J., *Ben-Asher and Isaac Yaesh, "Advances in Missile Guidance Theory"*, AIAA, 1998.

[9] *Yaakov Bar-Shalom, X.Rong Li, Thiaglaingam Kirubarajan, "Estimation with Applications to Tracking and Navigation"*, Wiley, 2001.