A Novel Multiplicative Quaternion Filter

Daniel Choukroun and Uri Tamir

Abstract This paper presents a novel quaternion filter from vector measurements that belongs to the realm of deterministic constrained least-squares estimation. Hinging on the interpretation of quaternion measurement errors as angular errors in a four-dimensional Euclidean space, a novel cost function is developed and a minimization problem is formulated under the quaternion unit-norm constraint. This approach sheds a new light on the Wahba problem and on the q-method. The optimal estimate can be interpreted as achieving the least angular distance among a collection of planes in $\mathbb{R}^4$ that are constructed from the vector observations. The resulting batch algorithm is mathematically equivalent to the q-method. Taking advantage of the gained geometric insight, a recursive algorithm is developed, where the update stage consists of a rotation in the four-dimensional Euclidean space. The rotation angle is empirically designed as a fading memory factor. The quaternion update stage is multiplicative thus preserving the estimated quaternion unit-norm and no iterative search for eigenvalues is required, as opposed to the q-method. The two algorithms are extended to the case of time varying attitude, under the assumptions that the inertial angular rates are measured. Extensive Monte-Carlo simulations showed that the proposed filter asymptotically converges to the q-method solution. The performances are numerically investigated for a range of typical values of the noise intensities in the rate gyroscopes and the attitude sensing.

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1 Introduction

This paper is concerned with the problem of attitude quaternion filtering from vector observations. The quaternion of rotation is popular for the purpose of attitude representation because it is a minimal non-singular attitude representation and the related rigid-body kinematics are described by a singularity-free linear ordinary differential equation [1, p. 411]. Over the last fifty years numerous quaternion estimators from vector observations were developed within the realm of deterministic constrained least-squares theory (see [4] for a review). That approach, known as the Wahba approach [2], lends itself to Davenport's q-method for quaternion single-frame estimation [1, p. 411], [3, Appendix A]. Single-frame batch algorithms were developed, where only the measurements acquired at a single time epoch were processed, discarding the past estimate [5, 6]. On the other hand recursive quaternion estimators were developed, extended to time-varying attitude, augmented in order to include parameters other than attitude, or optimized with respect to their gain [7, 8, 9, 10]. A feature that is common to this class of estimators is that the filtering part is performed on specific matrix quantities, namely the K-matrix or the B-matrix, and not on the quaternion per se. The drawbacks are that the filtered variable is of higher dimension than the quaternion, and that the constraint of unit-norm is imposed on the quaternion outside of the estimation process, that is, with no meaningful insight on its impact with respect to the estimation error. Furthermore, the filtered matrix variables do not lend themselves to a clear physical interpretation. In addition, these methods often rely on iterative eigenvector-eigenvalue solving steps.

This work's contribution is two-folds: first it consists in revisiting the Wahba problem and in providing an additional insight into the q-method. The proposed approach exploits a measurement model equation introduced in [11] and further investigated in [12]. The cornerstone of that model is that the sought quaternion lies in the Null space of a skew-symmetric matrix built from the vector observations. Based on that geometrical insight, a novel cost function is designed as the sum of the squared angular distances between the sought quaternion and the collection of planar Null spaces. Then a least-squares problem is formulated subject to the quaternion unit-norm constraint. The resulting batch quaternion estimator is shown to be mathematically equivalent to the q-method. The optimal quaternion is interpreted as the unique direction in $\mathbb{R}^4$ that achieves the least angular distance among the collection of two-dimensional Null spaces. The second contribution consists in developing a novel recursive filter of the time varying quaternion where the measurement update stage operates on the quaternion itself via a norm-preserving transformation in the Euclidean space $\mathbb{R}^4$. In the case of a time-varying attitude, the angular velocity is measured via rate gyroscopes with additive white noises. The performances of the proposed filter converge asymptotically to those of a sequential q-method.

The paper includes preliminary results in Section 2. Section 3 presents the formulation of the constrained least-squares quaternion estimation problem, and its equivalence with the Wahba problem and the q-method. Section 4 includes the development of the novel quaternion estimator for the single frame case. Section5 address the case of a time varying attitude. Section 6 presents a numerical investigation of the novel filter's performances. Section 7 presents the conclusions.

2 Preliminary Results

2.1 Quaternion Measurement Model

This section follows [12]. Consider a Cartesian coordinate frame $\mathscr{B}$ attached to a rigid body spacecraft, which is rotated with respect to a reference Cartesian coordinate frame $\mathscr{R}$. Let $\mathbf{b}$ and $\mathbf{r}$ denote the projections of a unit physical vector along the axes of $\mathscr{B}$ and $\mathscr{R}$, respectively, then

$$\mathbf{b} = D(\mathbf{q})\,\mathbf{r} \tag{1}$$

where $D(\mathbf{q})$ is the rotation matrix from $\mathscr{R}$ to $\mathscr{B}$, a.k.a. the attitude matrix of $\mathscr{B}$ with respect to $\mathscr{R}$. In Eq. (1), the vector $\mathbf{b}$ represents the true value of a normalized physical vector, like the Earth magnetic field or the line-of-sight to a celestial object, as observed in the spacecraft. Let the 3×1 vectors $\mathbf{s}$ and $\mathbf{d}$ and the 4×4 matrix H be defined as follows:

$$\mathbf{s} \triangleq \frac{1}{2}(\mathbf{b} + \mathbf{r}) \tag{2}$$

$$\mathbf{d} \triangleq \frac{1}{2}(\mathbf{b} - \mathbf{r}) \tag{3}$$

$$H \triangleq \begin{pmatrix} -[\mathbf{s}\times] & \mathbf{d} \\ -\mathbf{d}^T & 0 \end{pmatrix} \tag{4}$$

then, the attitude quaternion $\mathbf{q}$ associated with the rotation matrix $D(\mathbf{q})$ satisfies the following relationship

$$H\mathbf{q} = \mathbf{0} \tag{5}$$

Equation (5) is referred to as a (ideal) pseudo-measurement where the measurement is identically zero and the signal term is linear in $\mathbf{q}$. As a read-out of a noisy sensor, the quantity $\mathbf{b}$ is corrupted by an additive measurement noise, δb, which thus becomes multiplicative in $\mathbf{q}$, as follows:

$$H\mathbf{q} - \frac{1}{2}\,\Xi(\mathbf{q})\,\delta b = \mathbf{0} \tag{6}$$

where $\Xi(\mathbf{q})$ is a 4×3 linear matrix function of $\mathbf{q}$. It is possible to develop exact expressions for the first two moments of this multiplicative noise (see details in [12]). However, the scope of the present work is limited to a deterministic setting. Notice that Eq. (5) is different from the fundamental equation of the estimator ESOQ [Eq. (19), [6]], where the optimal estimate is sought to be in the Null space of a symmetric matrix.,

2.2 Pseudo-measurement Matrix Properties

An eigenvalue analysis of the pseudo-measurement matrix H yields

$$\det(\lambda \mathrm{I} - H) = \left(\lambda^2 + \frac{1}{2}\left(\|\mathbf{b}\|^2 + \|\mathbf{r}\|^2\right) + \frac{1}{16}\left(\|\mathbf{b}\|^2 - \|\mathbf{r}\|^2\right)^2\right)\lambda^2 \tag{7}$$

where $\|\mathbf{b}\|$ denotes the Euclidean norm of $\mathbf{b}$ in $\mathbb{R}^3$. Recalling that $\mathbf{b}$ and $\mathbf{r}$ are unit-norm vectors, the spectrum of H is as follows:

$$\mathrm{SpH} = \{0, 0, j, -j\} \tag{8}$$

Since H is skew-symmetric, there exist an orthonormal matrix $Q \in \mathbb{R}^{4\times 4}$ and a block diagonal matrix $\Lambda \in \mathbb{R}^{4\times 4}$, such that

$$\begin{aligned} H &= Q\Lambda Q^T \\ &= \begin{bmatrix}\mathbf{q}_1\, \mathbf{q}_2\, \mathbf{q}_3\, \mathbf{q}_4\end{bmatrix} \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{bmatrix} \begin{bmatrix}\mathbf{q}_1^T \\ \mathbf{q}_2^T \\ \mathbf{q}_3^T \\ \mathbf{q}_4^T\end{bmatrix} \end{aligned} \tag{9}$$

According to Eq. (5), the quaternion belongs to the Kernel of H, denoted by $\mathrm{Ker}H$, which, from Eq. (9), is a plane in $\mathbb{R}^4$ with orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2\}$. Analytical expressions for the complete basis of eigenvectors $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4\}$ of the matrix H are available as functions of the vectors $\mathbf{s}$ and $\mathbf{d}$ solely [12].

3 Single-Frame Batch Quaternion Estimation

3.1 Constrained Least-Squares Problem Formulation

Assume that a set of n vector measurements is acquired at a given epoch time, and that the associated matrices $\{H_k\}_{k=1,\dots,n}$ are computed, according to Eqs. (2) to (4). Ideally, they all are related to the quaternion $\mathbf{q}$ via Eq. (5). In practice, these measurements are corrupted by noises and the perturbations in the associated H-matrices induce perturbations in their Kernels. Henceforth, the quaternion does not necessarily belong to any of the "measured" $\mathrm{Ker}H_k$. For each noisy H_k, its matrix of eigenvectors, $Q(k)$, will be partitioned as follows

$$Q(k) = \begin{bmatrix}Q_{12}(k)\, Q_{34}(k)\end{bmatrix} \tag{10}$$

such that $Q_{12}(k)$ and $Q_{34}(k)$ are defined as

$$Q_{12}(k) \triangleq [\mathbf{q}_1\, \mathbf{q}_2] \tag{11}$$

$$Q_{34}(k) \triangleq [\mathbf{q}_3\, \mathbf{q}_4] \tag{12}$$

where the eigenvectors $\mathbf{q}_i$, $i = 1,2,3,4$ are computed from the noisy vectors $\mathbf{s}_k$ and $\mathbf{d}_k$ using Eqs. (**??**)-(**??**). Let $\mathbf{q}_k^*$ denote the normalized projection of the quaternion $\mathbf{q}$ on $\mathrm{Ker}\,H_k$:

$$\mathbf{q}_k^* = \frac{1}{(\mathbf{q}^T Q_{12}(k) Q_{12}^T(k)\mathbf{q})^{\frac{1}{2}}} Q_{12}(k) Q_{12}^T(k)\mathbf{q} \tag{13}$$

Let θ_k denote the angle between $\mathbf{q}$ and $\mathrm{Ker}\,H_k$, $0 \le \theta_k \le \frac{\pi}{2}$ (see Fig. 1), thus

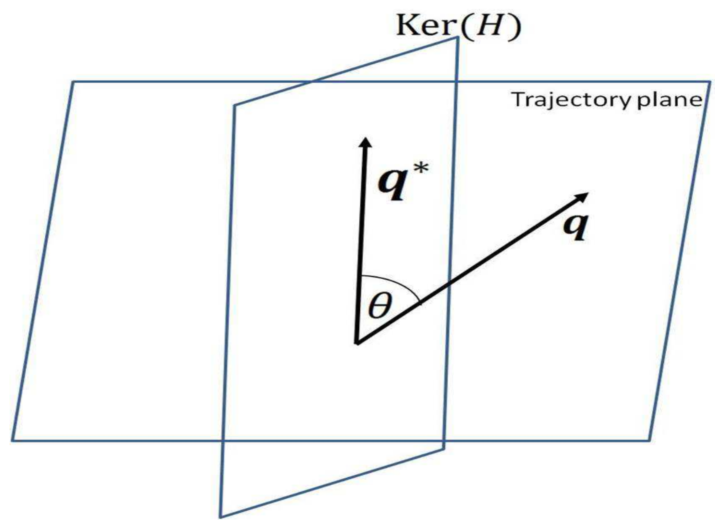

Fig. 1 $\mathbf{q}$ is the true quaternion and $\mathbf{q}^*$ is its normalized projection onto the plane $\mathrm{Ker}H$. The angular distance θ between the quaternion $\mathbf{q}$ and the plane $\mathrm{Ker}H$ is interpreted as a quaternion measurement error.

$$\cos\theta_k = \mathbf{q}^T \mathbf{q}_k^* \tag{14}$$

Using Eq. (13) in Eq. (14) yields

$$\cos\theta_k = (\mathbf{q}^T Q_{12}(k) Q_{12}^T(k)\mathbf{q})^{\frac{1}{2}} \tag{15}$$

If the vector measurements are ideal, then $\mathbf{q}$ will belong to the intersection of all the kernels, and the angular distances, as expressed by θ_k, would all be zero. In the case of noisy measurements, the angles θ_k provide meaningful interpretations of the measurement errors, and it seems adequate to seek for an estimate of $\mathbf{q}$ as the unique unit-norm vector that is the closest, in some sense, to the collection of planes

$\{\operatorname{Ker} H_k\}$, $k = 1,...,n$. Henceforth, the following estimation problem is formulated: *Given a set of n matrices* $\{H_k\}_{k=1,...,n}$*, computed from n pairs of vector observations* $\{\mathbf{b}_k, \mathbf{r}_k\}_{k=1,...,n}$ *as defined in Eqs.* (2)-(4)*, find the unit-norm vector* $\mathbf{q} \in \mathbb{R}^4$ *that solves*

$$\min_{\mathbf{q},\|\mathbf{q}\|=1} \sum_{k=1}^{n} \theta_k^2 \tag{16}$$

where θ_k *is defined from Eq.* (15)*, and* $\|\mathbf{q}\|$ *denotes the Euclidean norm in* $\mathbb{R}^4$.
The proposed loss function in Eq. (16) is interpreted as an angular distance between vector spaces. Thus the magnitude of the sought optimization vector, $\mathbf{q}$, does not change the cost value, so that scaling can be performed without loss of optimality. In the present case, the proper scaling is a normalization, in agreement with the rotation quaternion property. Notice that the freedom in choosing the sign of the solution does not yield any ambiguity since the two quaternions $\mathbf{q}$ and $-\mathbf{q}$ represent the same attitude. By convention, practitioners often restrict themselves to quaternions with positive scalar parts. Next, the constrained optimization problem (16) will be reformulated as follows. Since $\cos\theta_k$ is a decreasing function on $[0, \frac{\pi}{2}]$ the following equivalent problem is proposed

$$\max_{\mathbf{q},\|\mathbf{q}\|=1} \left\{ \sum_{k=1}^{n} (\cos\theta_k)^2 \right\} \tag{17}$$

Using Eq. (15) in Eq. (17) yields

$$\max_{\mathbf{q},\|\mathbf{q}\|=1} \left\{ \sum_{k=1}^{n} \mathbf{q}^T Q_{12}(k) Q_{12}^T(k) \mathbf{q} \right\} \tag{18}$$

Since $\mathbf{q}$ is not a function of k, Eq. (18) is equivalent to

$$\max_{\mathbf{q},\|\mathbf{q}\|=1} \left\{ \mathbf{q}^T \left[\sum_{k=1}^{n} Q_{12}(k) Q_{12}^T(k) \right] \mathbf{q} \right\} \tag{19}$$

Let M be the 4×4 matrix defined as

$$M \stackrel{\triangle}{=} \sum_{k=1}^{n} Q_{12}(k) Q_{12}^T(k) \tag{20}$$

then Eq. (19) is equivalent to

$$\max_{\mathbf{q},\|\mathbf{q}\|=1} \mathbf{q}^T M \mathbf{q} \tag{21}$$

Equation (21) describes a well known optimization problem, which is an "extremal characterization" of the maximal eigenvalue of the matrix M [14, p. 278]: the solution to Eq. (21) is the eigenvector that is associated with the maximal eigenvalue of the matrix M.

3.2 Relation with the q-method

Problem (21) is similar to Wahba's problem as formulated by Davenport using the quaternion [1, p. 411], [3, Appendix A]. The Wahba problem is formulated as follows:
Given a set of n single-frame vector measurements $\{\mathbf{b}_k\}_{k=1}^n$ *and* $\{\mathbf{r}_k\}_{k=1}^n$*, and the following loss function*

$$L(D) = \sum_{k=1}^{n} \| \mathbf{b}_k - D\mathbf{r}_k \|^2 \tag{22}$$

where D denotes the attitude matrix, and $\| \cdot \|$ *denotes the Euclidean norm in* $\mathbb{R}^3$*, solve the following constrained least-squares problem*

$$\min_D \left\{ \sum_{k=1}^{n} \| \mathbf{b}_k - D\mathbf{r}_k \|^2 \right\} \tag{23}$$

subject to D being a proper orthogonal matrix, i.e., $D^T D = I_3$ *and* $\det D = 1$.

The original Wahba loss function included positive scalar weights in the residuals terms, which are omitted here for the sake of simplicity, without loss of generality. Using the relation between the attitude matrix and the quaternion, Eq. (23), Davenport showed the equivalence of Eq. (23) with the following problem

$$\max_{\mathbf{q}, \|\mathbf{q}\|=1} \mathbf{q}^T K \mathbf{q} \tag{24}$$

where

$$K = \sum_{k=1}^{n} K_k \tag{25}$$

$$K_k = \begin{pmatrix} S_k - \sigma I_3 & \mathbf{z}_k \\ \mathbf{z}_k^T & \sigma_k \end{pmatrix} \tag{26}$$

$$S_k = B_k + B_k{}^T \tag{27}$$

$$B_k = \mathbf{b}_k \mathbf{r}_k^T \tag{28}$$

$$\mathbf{z}_k = \mathbf{b}_k \times \mathbf{r}_k \tag{29}$$

$$\sigma_k = \mathrm{Tr} B_k \tag{30}$$

In Eq. (30), "Tr" denotes the Trace operator. The q-method, hence, consists in computing the optimal quaternion as the eigenvector of the matrix K that is associated with the maximal eigenvalue. The following Proposition clarifies the relationship between the matrix K from Davenport's q-method and the matrix M introduced in Eq. (20).

Proposition:

Given a set of n single-frame vector measurements and the associated matrices K and M, as defined from Eqs. (25)-(30) *and Eq.* (20)*, respectively, then*

$$M = \frac{1}{2}\left(nI_4 + K\right) \tag{31}$$

The following conclusion can be drawn: the proposed constrained least-squares problem (16), or equivalently (21), and its solution are equivalent to the Wahba problem and the q-method, since the matrices M and K differ by a constant. This provides thus a novel insight on the q-method: the optimal quaternion is the unique direction in $\mathbb{R}^4$ which minimizes the angular distance to the set of planes, $\{\mathrm{Ker}\, H_k\}_{k=1,\ldots,n}$. The proof is omitted for the sake of brevity.

4 Single-Frame Recursive Quaternion Estimation

4.1 Filter of the M matrix

Given a new vector observation sample $(k+1)$, the cost function can be rewritten as follows:

$$\begin{aligned} J &= \mathbf{q}^T \left[\sum_{i=1}^{k+1}(I_4 - H_i^T H_i)\right]\mathbf{q} \\ &= \mathbf{q}^T \left[\sum_{i=1}^{k}(I_4 - H_i^T H_i) + (I_4 - H_{k+1}^T H_{k+1})\right]\mathbf{q} \end{aligned}$$

This shows that the M matrix at sample $(k+1)$ is calculated as follows:

$$M_{k+1} = M_k + \delta M_{k+1} \tag{32}$$

Equation (32) is a measurement update equation in a filter of the matrix M, where the “new” observation δM_{k+1} and the “previous” estimate M_k are weighted identically. The process is repeated as long as vector observations come in and the quaternion estimate is extracted only if needed. The latter requires implementing an eigenvalue/eigenvector solver.

4.2 A Novel Quaternion Filter Using the H matrix: the HQF

Hinging on the geometrical properties highlighted in the previous section, a different path is followed next in order to develop a quaternion recursive estimator, where the measurement update stage preserves by design the unit-norm property of the estimate. Let $\widehat{\mathbf{q}}_k$ denote the estimate of $\mathbf{q}$ at step k, which is computed from the k first measurements. Let H_{k+1} denote the pseudo-measurement matrix built from the $k+1^{st}$ vector observation, and $\mathbf{q}^*_{k+1}$ denote the normalized projection of $\widehat{\mathbf{q}}_k$ on $\mathrm{Ker}\, H_{k+1}$ (see Fig. 2). Thus, by construction,

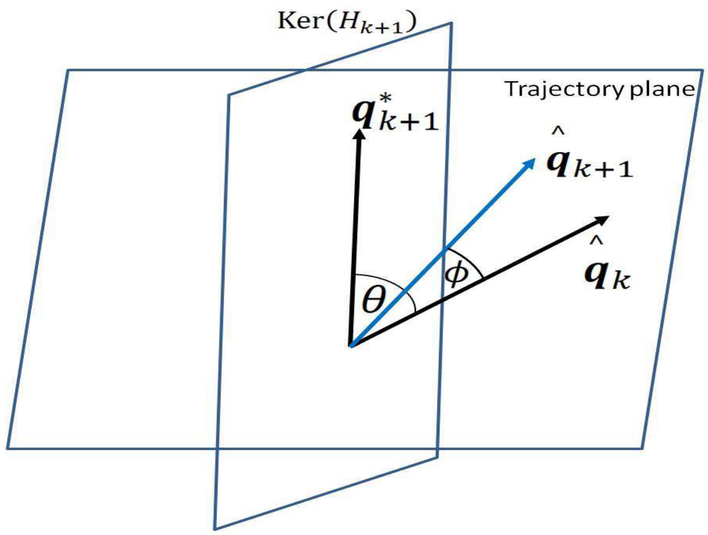

Fig. 2 Illustration of the relative geometry in $\mathbb{R}^4$ between the prior estimate, $\widehat{\mathbf{q}}_k$, its normalized projection on $\mathrm{Ker}\, H_{k+1}$, and the posterior estimate $\widehat{\mathbf{q}}_{k+1}$.

$$\mathbf{q}^*_{k+1} = \frac{1}{(\widehat{\mathbf{q}}_k^T Q_{12} Q_{12}^T \widehat{\mathbf{q}}_k)^{\frac{1}{2}}} Q_{12} Q_{12}^T \widehat{\mathbf{q}}_k \tag{33}$$

where the 4×2 matrix Q_{12} stems from the spectral decomposition of H_{k+1}. Let θ_k denote the angle between $\widehat{\mathbf{q}}_k$ and $\mathbf{q}^*_k$ in $\mathbb{R}^4$, then

$$(\cos\theta_k)^2 = \widehat{\mathbf{q}}_k^T Q_{12} Q_{12}^T \widehat{\mathbf{q}}_k \tag{34}$$

The measurement update stage of the proposed recursive algorithm consists in computing the updated estimate $\widehat{\mathbf{q}}_{k+1}$ via a rotation of $\widehat{\mathbf{q}}_k$ in the plane generated by $\left(\widehat{\mathbf{q}}_k, \mathbf{q}^*_{k+1}\right)$ and by an angle ϕ_k, where ϕ_k is parameterized as follows:

$$\phi_k = \alpha_k \theta_k \tag{35}$$

where $0 \leq \alpha_k \leq 1$ is a design parameter. The measurement update stage is thus expressed as follows

$$\widehat{\mathbf{q}}_{k+1} = A(\phi_k)\widehat{\mathbf{q}}_k \tag{36}$$

where $A(\phi_k)$ is an orthogonal matrix in $\mathbb{R}^4$. It, thus, preserves the unit norm of the quaternion estimate along the estimation process. The coefficient α_k has the function of a gain. If $\alpha_k = 0$, the update stage maintains the estimated quaternion at its current value, i.e. $\widehat{\mathbf{q}}_{k+1} = \widehat{\mathbf{q}}_k$. If $\alpha_k = 1$, the update stage generates the projection $\mathbf{q}_k^*$. With $\alpha_k = \frac{1}{k}$, for instance, there will be a fading memory effect as k increases. The next developments describe the construction of the matrix $A(\phi_k)$

4.3 Basis of the Rotation

For notational simplicity, the index k will be dropped in the following. Let $\left(\hat{\boldsymbol{I}},\hat{\boldsymbol{J}},\hat{\boldsymbol{K}},\hat{\boldsymbol{L}}\right)$ denote the following orthonormal basis vectors in $\mathbb{R}^4$ (see an illustration in Fig. 3).

$$\widehat{\boldsymbol{K}} = \mathbf{q}^* \tag{37}$$

$$\widehat{\boldsymbol{J}} \in \{\mathbf{q}^*, \widehat{\mathbf{q}}\}, \widehat{\boldsymbol{J}} \perp \widehat{\boldsymbol{K}} \tag{38}$$

$$\widehat{\boldsymbol{I}} \in \{\mathbf{q}_1, \mathbf{q}_2\}, \widehat{\boldsymbol{I}} \perp \widehat{\boldsymbol{K}} \tag{39}$$

$$\widehat{\boldsymbol{L}} \in \{\mathbf{q}_3, \mathbf{q}_4\}, \widehat{\boldsymbol{L}} \perp \widehat{\boldsymbol{J}} \tag{40}$$

where $\{\mathbf{q}^*, \widehat{\mathbf{q}}\}$ denotes the linear subspace generated by the set $\{\mathbf{q}^*, \widehat{\mathbf{q}}\}$, and $\perp$ denotes orthogonality in the Euclidian vector space $\mathbb{R}^4$; i.e $\mathbf{u} \perp \mathbf{v} \Longleftrightarrow \mathbf{u}^T \mathbf{v} = 0$. Thus, by definition,

$$\widehat{\boldsymbol{K}} = \frac{1}{(\widehat{\mathbf{q}}^T Q_{12} Q_{12}^T \widehat{\mathbf{q}})^{\frac{1}{2}}} Q_{12} Q_{12}^T \widehat{\mathbf{q}} \tag{41}$$

The basis vector $\hat{\boldsymbol{J}}$ is constructed via a Gram-Schmidt orthogonalization step

$$\bar{\boldsymbol{J}} = \widehat{\mathbf{q}} - ({\mathbf{q}^*}^T \widehat{\mathbf{q}}) \mathbf{q}^* \tag{42}$$

It is straightforward to show that

$$\bar{\boldsymbol{J}} = Q_{34} Q_{34}^T \widehat{\mathbf{q}} \tag{43}$$

Let $\hat{\boldsymbol{J}}$ be the normalized $\bar{\boldsymbol{J}}$, i.e.

$$\widehat{\boldsymbol{J}} = \frac{1}{(\widehat{\mathbf{q}}^T Q_{34} Q_{34}^T \widehat{\mathbf{q}})^{\frac{1}{2}}} Q_{34} Q_{34}^T \widehat{\mathbf{q}} \tag{44}$$

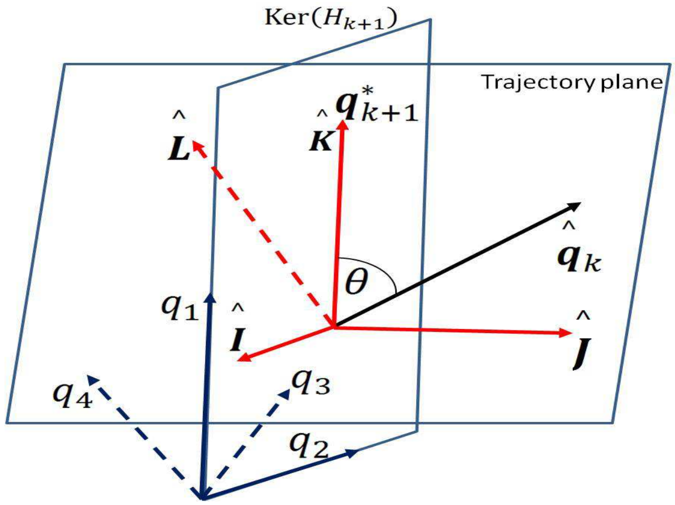

Fig. 3 The orthonormal basis $\left(\hat{\boldsymbol{I}},\hat{\boldsymbol{J}},\hat{\boldsymbol{K}},\hat{\boldsymbol{L}}\right)$ is used in order to perform a rotation in the trajectory plane generated by $\hat{\boldsymbol{J}}$ and $\hat{\boldsymbol{K}}$.

There are two possible vectors that satisfy the properties of $\widehat{\boldsymbol{I}}$, which differ by their sign. One of the two is defined below:

$$\bar{\boldsymbol{I}} = \mathbf{q}_1 - (\hat{\boldsymbol{K}}^T \mathbf{q}_1)\hat{\boldsymbol{K}} \tag{45}$$

Using Eq. (41) in Eq. (45) yields

$$\bar{\boldsymbol{I}} = \mathbf{q}_1 - (\widehat{\mathbf{q}}^T Q_{12} Q_{12}^T \widehat{\mathbf{q}})^{-1} (\widehat{\mathbf{q}}^T \mathbf{q}_1) Q_{12} Q_{12}^T \widehat{\mathbf{q}} \tag{46}$$

It stems from Eq. (46), using the orthogonality of $\mathbf{q}_1$ and $\mathbf{q}_2$, that

$$\begin{aligned} \|\bar{\boldsymbol{I}}\|^2 &= \frac{\widehat{\mathbf{q}}^T (\mathbf{q}_2 \mathbf{q}_2^T) \widehat{\mathbf{q}}}{\widehat{\mathbf{q}}^T Q_{12} Q_{12}^T \widehat{\mathbf{q}}} \\ &= \frac{(\widehat{\mathbf{q}}^T \mathbf{q}_2)^2}{\widehat{\mathbf{q}}^T Q_{12} Q_{12}^T \widehat{\mathbf{q}}} \end{aligned} \tag{47}$$

The unit vector $\hat{\boldsymbol{I}}$ is then defined as follows.

$$\widehat{\boldsymbol{I}} = \frac{1}{\|\bar{\boldsymbol{I}}\|} \bar{\boldsymbol{I}} \tag{48}$$

Using Eqs. (48) and (47) in (46) yields

$$
\begin{aligned}
\widehat{\boldsymbol{I}} &= \frac{(\widehat{\mathbf{q}}^T Q_{12} Q_{12}^T \widehat{\mathbf{q}})^{\frac{1}{2}}}{(\widehat{\mathbf{q}}^T \mathbf{q}_2)} \left(\frac{(\widehat{\mathbf{q}}^T Q_{12} Q_{12}^T \widehat{\mathbf{q}})\mathbf{q}_1 - (\widehat{\mathbf{q}}^T \mathbf{q}_1) Q_{12} Q_{12}^T \widehat{\mathbf{q}}}{\widehat{\mathbf{q}}^T Q_{12} Q_{12}^T \widehat{\mathbf{q}}} \right) \\
&= \frac{1}{(\widehat{\mathbf{q}}^T Q_{12} Q_{12}^T \widehat{\mathbf{q}})^{\frac{1}{2}}} \left(\frac{((\widehat{\mathbf{q}}^T \mathbf{q}_1)^2 + (\widehat{\mathbf{q}}^T \mathbf{q}_2)^2)\mathbf{q}_1 - (\widehat{\mathbf{q}}^T \mathbf{q}_1) Q_{12} Q_{12}^T \widehat{\mathbf{q}}}{\widehat{\mathbf{q}}^T \mathbf{q}_2} \right) \\
&= \frac{1}{(\widehat{\mathbf{q}}^T Q_{12} Q_{12}^T \widehat{\mathbf{q}})^{\frac{1}{2}}} \left((\widehat{\mathbf{q}}^T \mathbf{q}_2)\mathbf{q}_1 - (\widehat{\mathbf{q}}^T \mathbf{q}_1)\mathbf{q}_2 \right) \\
&= \frac{1}{(\widehat{\mathbf{q}}^T Q_{12} Q_{12}^T \widehat{\mathbf{q}})^{\frac{1}{2}}} Q_{12} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} Q_{12}^T \widehat{\mathbf{q}} \\
&= \frac{1}{(\widehat{\mathbf{q}}^T Q_{12} Q_{12}^T \widehat{\mathbf{q}})^{\frac{1}{2}}} Q_{12} \Lambda_1 Q_{12}^T \widehat{\mathbf{q}}
\end{aligned}
$$

Finally, let $\widehat{\boldsymbol{L}}$ be defined as follows

$$\bar{\boldsymbol{L}} = \mathbf{q}_3 - (\widehat{\boldsymbol{J}}^T \mathbf{q}_3)\widehat{\boldsymbol{J}} \tag{49}$$

Similar steps as those followed to define $\widehat{\boldsymbol{I}}$ are followed for $\widehat{\boldsymbol{L}}$, yielding

$$\widehat{\boldsymbol{L}} = \frac{1}{(\widehat{\mathbf{q}}^T Q_{34} Q_{34}^T \widehat{\mathbf{q}})^{\frac{1}{2}}} Q_{34} \Lambda_1 Q_{34} \widehat{\mathbf{q}} \tag{50}$$

Let $\mathscr{G}$ denote the orthonormal basis $(\hat{\boldsymbol{I}}, \hat{\boldsymbol{J}}, \hat{\boldsymbol{K}}, \hat{\boldsymbol{L}})$, and let G denote the following 4×4 matrix

$$G = \left[\widehat{\boldsymbol{I}}\ \widehat{\boldsymbol{J}}\ \widehat{\boldsymbol{K}}\ \widehat{\boldsymbol{L}}\right] \tag{51}$$

The matrix G is the transformation matrix from the orthonormal basis $\mathscr{G}$ to the canonical orthonormal basis in $\mathbb{R}^4$.

4.4 Rotation in the Trajectory Plane

In the ensuing, the plane generated by the basis vectors $\hat{\boldsymbol{J}}$ and $\hat{\boldsymbol{K}}$ will be referred to as the "trajectory plane". We wish to rotate $\widehat{\mathbf{q}}_k$ towards $\mathbf{q}_{k+1}^*$ by an angle ϕ_k . Let C_k denote the matrix of a rotation in $\mathbb{R}^4$ of angle ϕ_k that maintains the rotated vector within the plane $\{\hat{\boldsymbol{J}}, \hat{\boldsymbol{K}}\}$. In other words, this rotation keeps $\hat{\boldsymbol{I}}$ and $\hat{\boldsymbol{L}}$ invariant and is thus expressed as follows

$$C_k \triangleq \begin{bmatrix} 1 & . & . & . \\ . & \cos\phi_k & \sin\phi_k & . \\ . & -\sin\phi_k & \cos\phi_k & . \\ . & . & . & 1 \end{bmatrix} \tag{52}$$

Notice that the matrix C_k maps the components of a vector in the $\mathscr{G}$-basis to the the components of the rotated vector in the same basis.

4.5 Summary of the Novel Recursive Estimator

In order to rotate the estimated quaternion $\widehat{\mathbf{q}}_k$ in the trajectory plane, first a basis transformation is performed using the G matrix, in order to express the prior estimate, $\widehat{\mathbf{q}}_k$, in the basis $\mathscr{G}$. Then the resulting vector is rotated in the trajectory plane using the C_k matrix, as defined in Eq. (52). This yields the a posteriori estimate, yet expressed in the basis $\mathscr{G}$. Finally an inverse basis transformation is performed on that vector in order to express it in the canonical basis. The resulting algorithm is summarized as follows:
Given $\widehat{\mathbf{q}}_k$, when a new vector measurement $\left(\mathbf{b}_{k+1}, \mathbf{r}_{k+1}\right)$ is acquired, the pseudo-measurement matrix H_{k+1} and its eigenvectors system are computed using Eqs. (**??**)-(**??**). The angle θ_k is evaluated using

$$(\cos\theta_k)^2 = \widehat{\mathbf{q}}_k^T Q_{12} Q_{12}^T \widehat{\mathbf{q}}_k \tag{53}$$

If $\theta_k = 0$ then $\widehat{\mathbf{q}}_{k+1} = \widehat{\mathbf{q}}_k$.
If $\theta_k \neq 0$ then:
i. The projection $\mathbf{q}^*_{k+1}$ of $\widehat{\mathbf{q}}_k$ on $Ker(H_{k+1})$ is computed as well as the other basis vectors of $\mathscr{G}$, as follows

$$\widehat{\boldsymbol{I}} = \frac{1}{(\widehat{\mathbf{q}}^T Q_{12} Q_{12}^T \widehat{\mathbf{q}})^{\frac{1}{2}}} Q_{12} \Lambda_1 Q_{12}^T \widehat{\mathbf{q}} \tag{54}$$

$$\widehat{\boldsymbol{J}} = \frac{1}{(\widehat{\mathbf{q}}^T Q_{34} Q_{34}^T \widehat{\mathbf{q}})^{\frac{1}{2}}} Q_{34} Q_{34}^T \widehat{\mathbf{q}} \tag{55}$$

$$\widehat{\boldsymbol{K}} = \frac{1}{(\widehat{\mathbf{q}}^T Q_{12} Q_{12}^T \widehat{\mathbf{q}})^{\frac{1}{2}}} Q_{12} Q_{12}^T \widehat{\mathbf{q}} = \mathbf{q}^* \tag{56}$$

$$\widehat{\boldsymbol{L}} = \frac{1}{(\widehat{\mathbf{q}}^T Q_{34} Q_{34}^T \widehat{\mathbf{q}})^{\frac{1}{2}}} Q_{34} \Lambda_1 Q_{34}^T \widehat{\mathbf{q}} \tag{57}$$

where

$$\Lambda_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{58}$$

and the basis transformation matrix, G_k, is computed as follows

$$G_k = \left[\widehat{\boldsymbol{I}}\ \widehat{\boldsymbol{J}}\ \widehat{\boldsymbol{K}}\ \widehat{\boldsymbol{L}}\right] \tag{59}$$

ii. The coefficient α_k is chosen via numerical simulations, and the rotation angle and matrix are computed as follows:

$$0 \leq \alpha_k \leq 1 \tag{60}$$

$$\phi_k = \alpha_k \theta_k \tag{61}$$

$$C_k = \begin{bmatrix} 1 & . & . & . \\ . & \cos\phi_k & \sin\phi_k & . \\ . & -\sin\phi_k & \cos\phi_k & . \\ . & . & . & 1 \end{bmatrix} \tag{62}$$

iii. The measurement update matrix A_k is computed as

$$A_k = G_k C_k G_k^T \tag{63}$$

and the quaternion estimate is updated as

$$\widehat{\mathbf{q}}_{k+1} = A_k \widehat{\mathbf{q}}_k \tag{64}$$

Since the matrices G_k and C_k are, by design, orthonormal matrices, the update matrix A_k is also orthonormal. This property ensures the preservation of the norm of the quaternion estimate along the estimation process.

5 Time Varying Quaternion Estimation

5.1 Filter of the M matrix

The quaternion kinematics equation is written as follows:

$$\mathbf{q}_{k+1} = \Phi_k \mathbf{q}_k \tag{65}$$

where Φ_k is a matrix exponential function of the angular velocity vector of the body frame with respect to the inertial frame, expressed in the body frame. The matrix Φ_k is a four dimensional orthogonal matrix, ie $\Phi_k^{-1} = \Phi_k^T$. The latter property as follows in order to develop the propagation step of the filter for the M matrix. Let $M_{k/k}$ denote the M matrix related to the attitude at time k and calculated using all vector observations until time k. Similarly, let $M_{k+1/k}$ denote the M matrix related to the attitude at time $k+1$ and calculated using all vector observations until time k. Consider the cost function of the quaternion $\mathbf{q}_k$, that is

$$J = \mathbf{q}_k{}^T M_{k/k} \mathbf{q}_k \tag{66}$$

and rewrite it as a function of the quaternion $\mathbf{q}_{k+1}$ thanks to the kinematics equation Eq. (65) and to the orthogonal property of Φ_k, that is

$$J = \mathbf{q}_{k+1}^T \left(\Phi_k M_{k/k} \Phi_k^T \right) \mathbf{q}_{k+1} \tag{67}$$

As a result, the following time propagation stage for the M matrix can be identified:

$$M_{k+1/k} = \Phi_k M_{k/k} \Phi_k^T \tag{68}$$

The quaternion $\mathbf{q}_{k+1/k}$ can then extracted, if needed, from $M_{k+1/k}$. Notice that Eq. (68) is a similarity transformation on the matrix M and thus induces a linear transformation on the eigenvectors of that matrix as follows:

$$\widehat{\mathbf{q}}_{k+1/k} = \Phi_k \widehat{\mathbf{q}}_{k/k} \tag{69}$$

5.2 Time Varying HQF

Borrowing from Eq. (69), and by analogy with the standard quaternion filters, the following step is added as the time propagation step.

$$\widehat{\mathbf{q}}_{k+1/k} = \Phi_k \widehat{\mathbf{q}}_{k/k} \tag{70}$$

6 Numerical Simulation

The gain factor, α_k, is chosen as $1/k$, where k denotes the current number of measurement samples. The rational behind this choice stems from the widespread approach of fading out the impact of the incoming measurement with respect to the prior estimate as the number of samples grows. The proposed algorithm is verified via Monte-Carlo (MC) simulations. In each MC run the initial true quaternion is set at random by generating its components from a uniform probability distribution with support [-1,1] and normalizing the resulting vector. The kinematics of the true quaternion are propagated using low and high inertial angular rates, ie $\omega = 0.1\,[1,1,1]\,[deg/sec]$ and $\omega = 90\,[1,1,1]\,[deg/sec]$, respectively. The propagation time step Δt is 0.1 sec. The angular rates are measured by a rate gyroscope with zero-mean additive white noises of intensity $\sigma_{eps}^2/\Delta t$, whose values range from very high quality, 0.01 $deg/\sqrt{sec}$ to very low quality, 1 $deg/\sqrt{sec}$. The true vector observations are acquired with a sample time Δ_{upd} of 1 second and generated at random. The measurements are simulated by adding zero mean white noises with intensity σ_b^2/Δ_{upd} to each component and normalizing the resulting vector. The values of σ_b range from star-tracker accuracy (0.01 deg) to coarse Sun sensor accuracy (10 deg). The proposed filter, denoted here HQF, is implemented as well as another recursive filter, Optimal REQUEST that is essentially an optimized recursive implementation of the q-method. Each MC run lasts 150 seconds. The figures of merit used in the following are the MC average and standard deviation of the angle between the true and the estimated quaternion at the final time, denoted by $\delta\phi_f$,. Table 1 features the values of the MC averages of $\delta\phi_f$ with the MC standard deviations appearing in parentheses. The results clearly indicate a robust behavior of the filter in the wide range of potential noise intensities, along with the consistent increase of performances for increasing accuracies. In particular, when the gyro noise is very small, the filter yields a tenfold improvement in the accuracy compared to the vector measurements. Figure 4 compares the performances of the HQF with the OPREQ filter for low angular rates in two cases: very accurate and very noisy sensors corresponding to $\sigma_b = 0.01[deg], \sigma_\varepsilon = 0.001[deg/\sqrt{sec}]$ for the upper graph and $\sigma_b = 10[deg], \sigma_\varepsilon = 1[deg/\sqrt{sec}]$ for the lower graph. For the sake of fairness, both filters are initialized identically, ie after the first two measurements the OPREQ quaternion estimate is provided as initial condition to the HQF. Then the two algorithms progress on their own. It is seen that OPREQ is performing slightly better than the HQF filter. Yet the HQF tends to the same level of accuracy as OPREQ while avoiding the burden of an eigenvalue/eigenvector solver at each step. The latter is clearer for low angular rates (upper graph) than for high angular rates (lower graph). Figures 5 and 6 depict the time histories of the MC average and of the $\pm 1\sigma$ envelopes of $\delta\phi$ for excellent and poor sensors accuracies, respectively. In each figure the upper and lower plots correspond to low and high angular rates, respectively. For very accurate sensors higher angular rates have a negative impact on the steady-state level of the angular error. For very noisy sensors on the other hand the impact of the angular rates seems marginal.

Table 1 Performances of the HQF at the final time. 100 MC runs

σ_b $[deg]$ / $\sigma_\varepsilon [\frac{deg}{\sqrt{\sec}}]$	0.01	0.1	1	10
0.001	(0.0006) 0.002	(0.006) 0.02	(0.04) 0.1	(0.3) 1.5
0.01	(0.003) 0.01	(0.008) 0.02	(0.07) 0.14	(0.3) 1.6
0.1	(0.03) 0.13	(0.07) 0.17	(0.11) 0.26	(0.98) 1.8
1	(0.8) 1.6	(0.9) 1.9	(1.1) 2.1	(1.1) 2.3

7 Conclusion

In this paper a novel time varying quaternion estimator from vector observations was developed within the realm of deterministic constrained least-squares estimation. A loss function was presented based on the property that the quaternion of rotation lies close to the Kernel planes of matrices constructed from the vector measurements. The proposed problem was shown to be mathematically equivalent to Davenport's q-method yielding a new insight on the Wahba problem. Indeed the optimal quaternion is the unique direction in $\mathbb{R}^4$ which minimizes the angular distance to the set

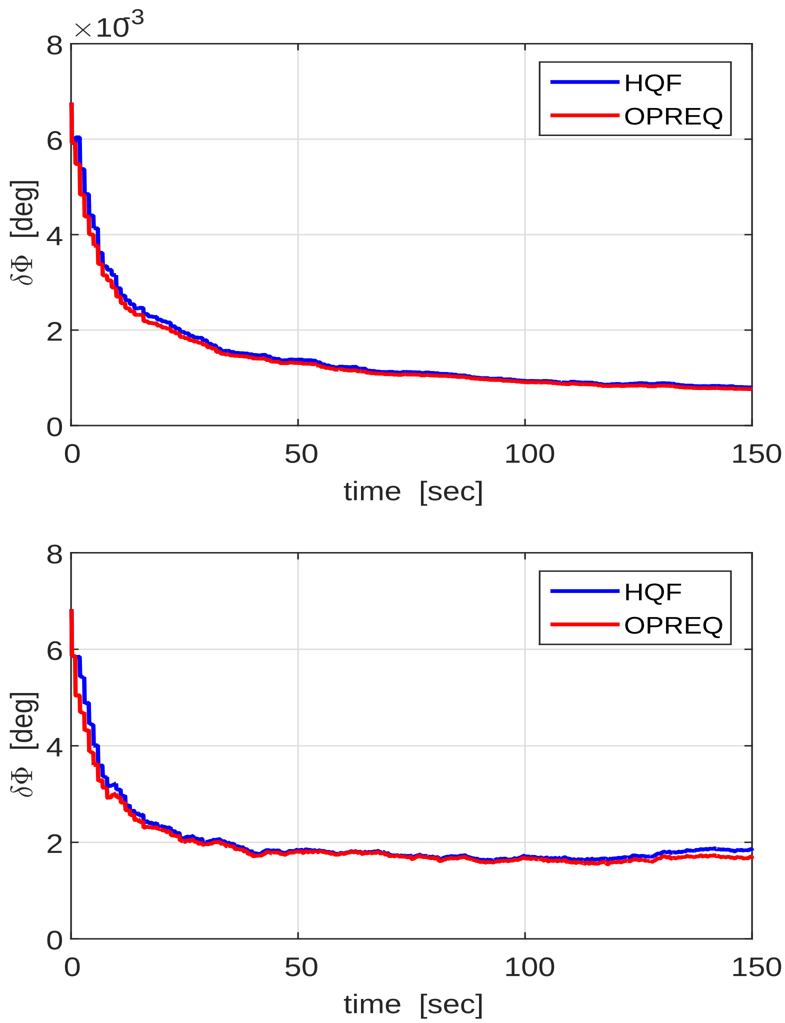

Fig. 4 Monte-Carlo averages of the angular estimation error (100 runs). Low angular rates. Upper graph-Very accurate sensors. Lower graph-Very noisy sensors. The performances of the HQF and of the recursive q-method converge asymptotically.

of the Kernel planes. In addition, a recursive algorithm for time varying quaternion estimation was developed based on this insight. The algorithm update stage features a rotation of the estimated quaternion in the Euclidean space $\mathbb{R}^4$. The rotation maintains the unit norm property of the quaternion and finds an adequate weighting between the current estimate and its normalized projection onto the most recent Kernel plane. A numerical simulation was performed showing that the proposed recursive algorithm exhibit similar asymptotic performances as the sequential q-method. This

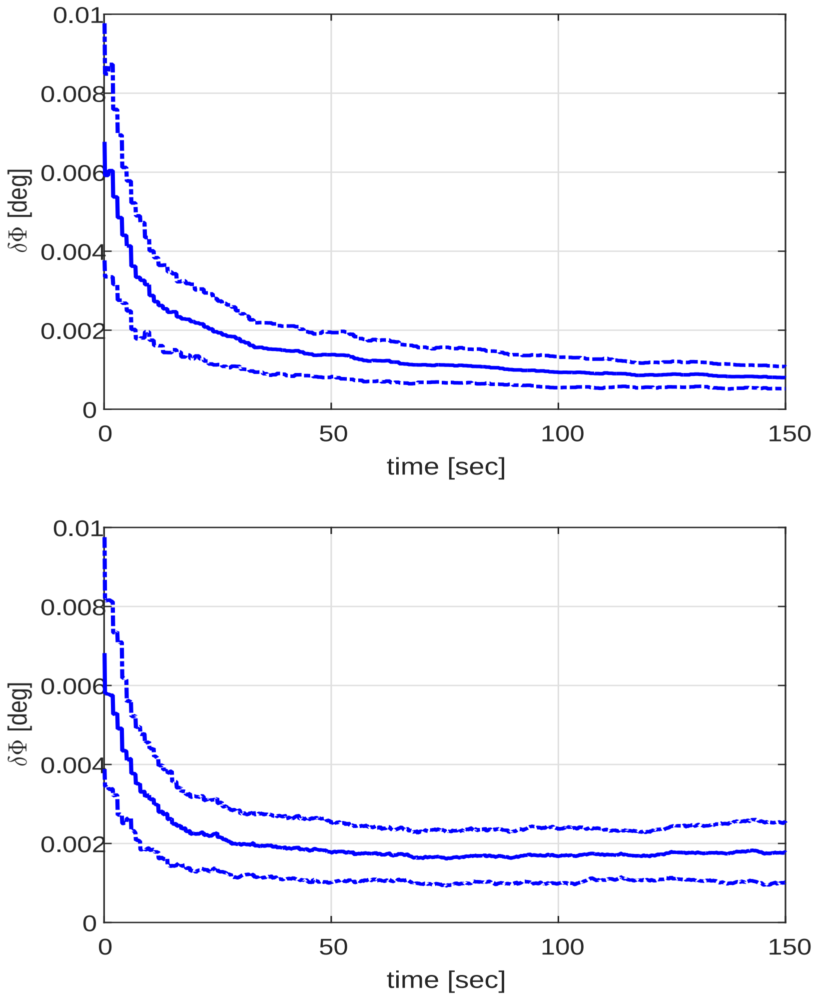

Fig. 5 Monte-Carlo averages and $\pm 1\sigma$ envelope of the angular estimation error (100 runs). Very accurate sensors. Upper graph-Low angular rates. Lower graph-High angular rates. The performances of the HQF are impacted by the high angular rates.

however is achieved at a computational lesser price than the q-method, where an iterative eigenvalue/eigenvector solver is implemented. This promising result was obtained while applying a fading-memory approach in the filter gain. Further work will focus on a systematic design of the filter gain α_k based on a statistical error analysis.

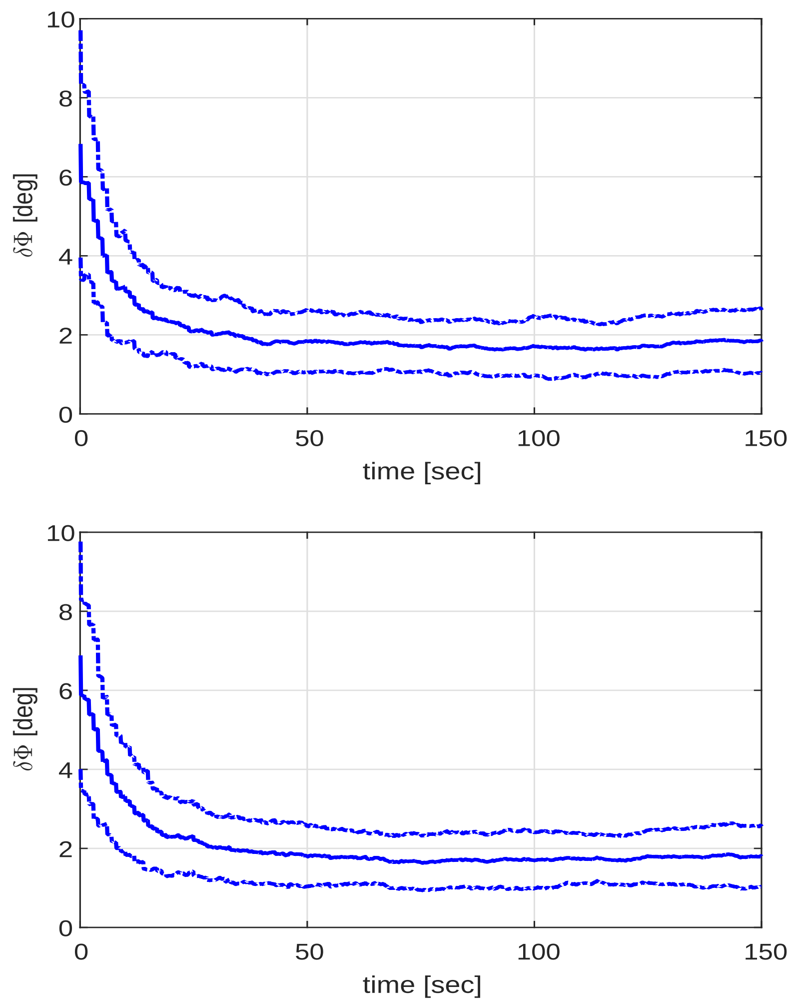

Fig. 6 Monte-Carlo averages and $\pm 1\sigma$ envelope of the angular estimation error (100 runs). Very noisy sensors. Upper graph-Low angular rates. Lower graph-High angular rates. The impact of the angular rates on the performances of the HQF is marginal.

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